Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Forms of Rolle's Theorem.

By J. C. Glashan, *Ottawa*, *Canada*.

CONTENTS.

Explanation of Symbols.

I. *Functions of a Simple Variable.* 1. Fundamental Form (Rolle's). 2. Extension of Rolle's Form. 3. Lagrange's Form. 4. Cauchy's Form. 5. Special case of Cauchy's Form. 6. Extension of Cauchy's Form. 7. Forms expressed by definite integrals. 8. Remainder in Taylor's Theorem. 9. Remainder in Cayley's Theorem. 10. Remainder in Lagrange's Series.

II. *Functions of a Complex Variable.* 11. General Form. 12. Special Form. 13. Remainder in Taylor's Theorem.

Explanation of Symbols.

$$0 < \theta < 1. \qquad a < \mu < b.$$

$$a\,;{}_m^n \equiv a_m + a_{m+1} + \ldots\ldots + a_{n-1}. \qquad a\,;n \equiv a\,;{}_0^n. \qquad \alpha_m \equiv \theta_m a_m.$$

$$\left[\int_0^a da\right]_m^{;\,n} U \equiv \int_0^{a\,;\,m+1} d(a\,;m+1) \int_0^{a\,;\,m+2} d(a\,;\,m+2) \ldots \int_0^{a\,;\,n} d(a\,;n)\,U.$$

$$\left[\int_0^a da\right]^{;\,n} U \equiv \left[\int_0^a da\right]_0^{;\,n} U.$$

$$[a]_m^n \equiv (n-m)!\left[\int_0^a da\right]_m^{;\,n} = \text{the remainder of the expansion of } (a\,;n)^{n-m}$$

after rejecting all terms of the form

$$C(a\,;m+1)^{n_m}\, a_{m+1}^{n_{m+1}}\, a_{m+2}^{n_{m+2}} \ldots\ldots\ldots\ldots,$$

in which

$$\begin{aligned} n_{m+1} + n_{m+2} + \ldots\ldots\ldots\ldots + n_{n-1} &> n-(m+1), \\ n_{m+2} + \ldots\ldots\ldots\ldots + n_{n-1} &> n-(m+2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ n_{n-1} &> 1. \end{aligned}$$

$$[a]^n \equiv [a]_0^n \equiv n!\left[\int_0^a da\right]^{;\,n}.$$

If u be a complex variable,

$\mathfrak{S}fu \equiv$ the protensive (real) part of fu;
$\mathfrak{V}fu \equiv$ the ditensive (imaginary) part of fu.

I. FUNCTIONS OF A SIMPLE VARIABLE.

1. *Fundamental Form.* (*Rolle's.*) Let Fx and $F'x$ be continuous from $x = a$ to $x = b$, and $Fb - Fa = 0$; then will

$$F'\mu = 0. \qquad \text{i.}$$

A proof of this theorem is given in nearly every text-book on the differential calculus; perhaps the best is that by Professor Mansion in his *Leçons d'Analyse infinitésimale*, (*Gand*).

2. *Extension of Rolle's Form.* If fx, ϕx and $f'x : \phi'x$ be continuous from $x = a$ to $x = b$, and $fb - fa = 0$, then will

$$f'\mu : \phi'\mu = 0. \qquad \text{ii.}$$

This form may be proved by reasoning similar to that employed to prove the Fundamental Form, or it may be deduced from the latter thus:

Let $fx = F\phi x$ and $\mu_1 \equiv \phi a + \theta\,(\phi b - \phi a)$; then, since fx and ϕx are both continuous from $x = a$ to $x = b$, $F\phi x$ will remain continuous while ϕx varies *continuously* from ϕa to ϕb, also $D_{\phi x}F\phi x = D_x F\phi x : D_x\phi x = f'x : \phi'x$ is continuous between the same limits.

Therefore by i. $$D_{\mu_1}F\mu_1 = 0 \qquad (\alpha).$$

But since ϕx is continuous from $x = a$ to $x = b$, although it need not always be intermediate between ϕa and ϕb, yet it must be capable of assuming any proposed intermediate value for at least one value of x between a and b, *i. e.* $\mu_1 = \phi\mu$.

Hence $D_{\mu_1}F\mu_1 = D_{\phi\mu}F\phi\mu = D_\mu F\phi\mu : D_\mu\phi\mu = f'\mu : \phi'\mu, = 0$ by (α).

3. *Lagrange's Form.* Let F_1x and $F_1'x$ be continuous from $x = a$ to $x = b$, then will

$$F_1b - F_1a = (b - a)\,F_1'\mu. \qquad \text{iii.}$$

Let $$Fx \equiv \frac{F_1x}{F_1b - F_1a} - \frac{x}{b - a},$$

then $$F'\mu \equiv \frac{F_1'\mu}{F_1b - F_1a} - \frac{1}{b - a} = 0, \text{ by i.};$$

hence $$F_1b - F_1a = (b - a)\,F_1'\mu.$$

4. *Cauchy's Form.* Let f_1x and ϕ_1x and either, $(A,)$ $f_1'x$ and ϕ'_1x, $(\phi'_1\mu \gtrless 0)$ or, $(B,)$ $f_1'x : \phi_1'x$ be continuous from $x = a$ to $x = b$, then will

$$f_1b - f_1a = \frac{f_1'\mu}{\varphi_1'\mu}(\phi_1b - \phi_1a). \qquad \text{iv.}$$

Case A. Let

$$Fx \equiv \frac{f_1x}{f_1b - f_1a} - \frac{\varphi_1x}{\varphi_1b - \varphi_1a},$$

then

$$F'\mu \equiv \frac{f_1'\mu}{f_1b - f_1a} - \frac{\varphi_1'\mu}{\varphi_1b - \varphi_1a} = 0, \text{ by i.};$$

whence, since $\phi_1'\mu \gtrless 0$,

$$f_1b - f_1a = \frac{f_1'\mu}{\varphi_1'\mu}(\phi_1b - \phi_1a). \qquad \text{iv}_1.$$

Case B. Let

$$fx \equiv \frac{f_1x}{f_1b - f_1a} - \frac{\varphi_1x}{\varphi_1b - \varphi_1a},$$

then

$$\frac{f'\mu}{\varphi_1'\mu} \equiv \frac{f_1'\mu}{\varphi_1'\mu} \cdot \frac{1}{f_1b - f_1a} - \frac{1}{\varphi_1b - \varphi_1a} = 0, \text{ by ii.};$$

whence

$$f_1b - f_1a = \frac{f_1'\mu}{\varphi_1'\mu}(\phi_1b - \phi_1a). \qquad \text{iv}_2.$$

This is an analytical translation of the geometrical proof given by Professor Mansion in his *Leçons*, p. 24, and also in *Messenger of Mathematics*, V_2, 34–35. The analytical demonstrations usually given in the text-books apply only to *Case A*.

5. *Special Case of Cauchy's Form.* Let U_x, V_x, W_x, and either D_xU_x and $D_x(V_x - W_x)$, $[D_\mu(V_\mu - W_\mu) \gtrless 0]$, or $D_xU_x : D_x(V_x - W_x)$ be all continuous from $x = a$ to $x = b$, then if

$$U_x \equiv f(e+x) + \frac{g-x}{1}f'(e+x) + \ldots\ldots\ldots + \frac{(g-x)^n}{n!}f^n(e+x),$$

$$V_x \equiv \phi(\varepsilon+x) + \frac{l-x}{1}\phi'(\varepsilon+x) + \ldots\ldots\ldots + \frac{(l-x)^p}{p!}\phi^p(\varepsilon+x),$$

$$W_x \equiv \psi(\eta-x) + \frac{x-\lambda}{1}\psi'(\eta-x) + \ldots\ldots\ldots + \frac{(x-\lambda)^q}{q!}\psi^q(\eta-x),$$

it follows immediately from iv. that

$$U_b - U_a = (V_b + W_a - V_a - W_b)\frac{p!\,q!\,(g-\mu)^n f^{n+1}(e+\mu)}{n!\{q!(l-\mu)^p\varphi^{p+1}(\varepsilon+\mu) + p!(\mu-\lambda)^q\psi^{q+1}(\eta-\mu)\}}. \qquad \text{v.}$$

For

$$D_xU_x \equiv \frac{(g-x)^n}{n!}f^{n+1}(e+x), \qquad D_xV_x \equiv \frac{(l-x)^p}{p!}\phi^{p+1}(\varepsilon+x),$$

and

$$D_xW_x \equiv -\frac{(x-\lambda)^q}{q!}\psi^{q+1}(\eta-x).$$

6. *Extension of Cauchy's Form.* Let f_1x, $f_1'x$, f_2x, $f_2'x$, ... f_nx, $f_n'x$, ϕ_1x, $\phi_1'x$, ϕ_2x, $\phi_2'x$, ... ϕ_nx, ϕ'_nx be all continuous from $x=a$ to $x=b$, then will

$$\frac{f'_1\mu}{f_1b-f_1a}+\frac{f'_2\mu}{f_2b-f_2a}+\,.\;.\;.\;.\;.\;.\;.\;.\,+\frac{f'_n\mu}{f_nb-f_na}$$
$$-\left\{\frac{\varphi'_1\mu}{\varphi_1b-\varphi_1a}+\frac{\varphi'_2\mu}{\varphi_2b-\varphi_2a}+\,.\;.\;.\;.\;.\,+\frac{\varphi'_n\mu}{\varphi_nb-\varphi_na}\right\}=0. \qquad \text{vi.}$$

Let
$$Fx\equiv\frac{f_1x}{f_1b-f_1a}+\frac{f_2x}{f_2b-f_2a}+\,.\;.\;.\;.\;.\,+\frac{f_nx}{f_nb-f_na}$$
$$-\left\{\frac{\varphi_1x}{\varphi_1b-\varphi_1a}+\frac{\varphi_2x}{\varphi_2b-\varphi_2a}+\,.\;.\;.\;.\;.\,+\frac{\varphi_nx}{\varphi_nb-\varphi_na}\right\},$$

then i. becomes vi.

Cor. If $fx=f_1x=f_2x=\,.\;.\;.\;.\;.\;.\,=f_nx$, then will

$$fb-fa=nf'\mu:\left\{\frac{\varphi'_1\mu}{\varphi_1b-\varphi_1a}+\,.\;.\;.\;.\;.\;.\;.\;.\,+\frac{\varphi'_n\mu}{\varphi_nb-\varphi_na}\right\}. \qquad \text{vii.}$$

Fx being continuous, n must in general be finite.

7. *Forms expressed by Definite Integrals.* Let fx, ϕx, ψx, $\int_c^x fx.\phi xdx$ and $\int_c^x fx.\psi xdx$ be continuous from $x=a$ to $x=b$, and $fx \geqq 0$ and $\psi x \geqq 0$, then will

$$\int_a^b fx.\phi xdx=\frac{\varphi\mu}{\psi\mu}\int_a^b fx.\psi xdx. \qquad \text{viii.}$$

Let
$$Fx\equiv\frac{\int_c^x fx.\phi xdx}{\int_a^b fx.\phi xdx}-\frac{\int_c^x fx.\psi xdx}{\int_a^b fx.\psi xdx};$$

then
$$F'\mu\equiv\frac{f\mu.\phi\mu}{\int_a^b fx.\phi xdx}-\frac{f\mu.\psi\mu}{\int_a^b fx.\psi xdx}=0 \text{ by i.}$$

consequently
$$\int_a^b fx.\phi xdx=\frac{\varphi\mu}{\psi\mu}\int_a^b fx.\psi xdx.$$

Cor. If $\psi x=1$, then will

$$\int_a^b fx.\phi xdx=\phi\mu\int_a^b fxdx. \qquad \text{ix.}$$

In viii. and ix. let $\phi x \equiv \int_c^x \phi_1 y dy$ and $\psi x \equiv \int_c^x \psi_1 y dy$

viii. becomes
$$\int_a^b fx \int_c^x \phi_1 y dy dx = \frac{\int_c^\mu \phi_1 x dx}{\int_c^\mu \psi_1 x dx} \int_a^b fx \int_c^x \psi_1 y dy dx; \qquad \text{x.}$$

ix. becomes
$$\int_a^b fx \int_c^x \phi_1 y dy dx = \int_a^b fx dx \,.\, \int_c^\mu \phi_1 x dx \,. \qquad \text{xi.}$$

Similarly
$$\int_a^b fx \int_c^x \phi y \int_e^y \psi z dz dy dx = \int_a^b fx dx \,.\, \int_c^\mu \phi x dx \,.\, \int_e^{\mu_1} \psi x dx \qquad \text{xii.}$$

where
$$\mu_1 = c + \theta(\mu - c)\,.$$

Let $\phi' x \equiv \int \phi x dx$ and let $\phi' c$ be finite. Integrating by parts,

$$\int_a^b fx \,.\, \phi x dx = fb \,.\, \phi' b - fa \,.\, \phi' a - \int_a^b f'x \,.\, \phi' x dx$$

$$= fb(\phi' b - \phi' c) - fa(\phi' a - \phi' c) - \int_a^b f'x (\phi' x - \phi' c)\, dx$$

$$= fb \int_c^b \phi x dx - fa \int_c^a \phi x dx - \int_a^b f'x \int_c^x \phi y dy dx \qquad (\beta).$$

by xi.,
$$= fb \int_c^b \phi x dx - fa \int_c^a \phi x dx - (fb - fa) \int_c^\mu \phi x dx \,. \qquad \text{xiii.}$$

Let $c = a$, then

$$\int_a^b fx \,.\, \phi x dx = fb \int_a^b \phi x dx - \int_a^b f'x \int_a^x \phi y dy dx \qquad (\gamma).$$

$$= fb \int_a^b \phi x dx - (fb - fa) \int_a^\mu \phi x dx \,. \qquad \text{xiv.}$$

xiv. may also be obtained directly from ix., thus:

$$\int_a^b fx \,.\, \phi x dx = fb \,.\, \phi' b - fa \,.\, \phi' a - \int_a^b f'x \,.\, \phi' x dx$$

by ix.,
$$= fb \,.\, \phi' b - fa \,.\, \phi' a - (fb - fa)\, \phi' \mu$$

$$= fb \int_\mu^b \phi x dx + fa \int_a^\mu \phi x dx \qquad \text{xiv}_1.$$

and
$$= fb \int_a^b \phi x dx - (fb - fa) \int_a^\mu \phi x dx \qquad \text{xiv.}$$

and also
$$= fa \int_a^b \phi x dx + (fb - fa) \int_\mu^b \phi x dx \,. \qquad \text{xiv}_2.$$

The usual proofs of viii. and ix. are substantially the same as Moigno's proof of Cauchy's Theorem, but no hint is at the same time given that iii. and viii. differ only in form of statement. xi., for the case $c = b$, was given by A. Winckler in *Sitzungsberichte der math.-nat. Klasse*, Wien, LX, (1869). (γ) was given by Grunert in *Grunert's Archiv.*, IV, 113. (β) was given by U. H. Meyer in *Grunert's Archiv.*, V, 216. xiv., xiv_1, and xiv_2 were given by Hankel in *Schlömilch's Zeitschrift*, XIV, (1869).

8. *Remainder in Taylor's Theorem.* In the Special Case of Cauchy's Form, let $g = l = b$, $\lambda = a$, $e = o$, $\varepsilon = c - a$, $\eta = k + b$ and $b = a + h$, then v. becomes

$$f(a+h) - \left\{ fa + \frac{h}{1} f'a + \ldots\ldots + \frac{h^n}{n!} f^n a \right\}$$

$$= \Big[\phi(c+h) - \left\{ \phi c + \frac{h}{1} \phi' c + \ldots\ldots + \frac{h^p}{p!} \phi^p c \right\}$$

$$+ \psi(k+h) - \left\{ \psi k + \frac{h}{1} \psi' k + \ldots\ldots + \frac{h^q}{q!} \psi^q k \right\} \Big]$$

$$\times \frac{p!\, q!\, (1-\theta)^n h^n f^{n+1}(a+\theta h)}{n!\,[q!\,(1-\theta)^p h^p \varphi^{p+1}(c+\theta h) + p!\, \theta^q h^q \psi^{q+1}\{k + (1-\theta)h\}]} \equiv R. \qquad \text{xv.}$$

R gives at once all the forms, exclusive of those involving definite integrals, hitherto proposed for the remainder in Taylor's Theorem for a simple variable. Thus, if ψx be constant and $c = a$, R will become the *general* form given in the *Mémoires de l'Académie ..... de Montpellier*, V. (1861–1863), and if in addition $p = 0$, R will become the *special* form given in the same memoir. (*Todhunter's Dif. Calc.*, 6th *edition*, pp. 404–406). The latter form is also given by Schlömilch in his *Uebungsbuch*, p. 262. If $p = n$, R will become the form given by Professor Mansion in *Messenger of Math.*, V_2, 161.

If ϕx be constant, R will become a form of which the special case for $k = q = 0$ was given by Schlömilch in *Liouville's Journal*, III_2, 384, (1858); also in his *Handbuch*, 1847. The forms of Roche (1858), Cauchy, and Lagrange (1807) are all particular cases of this form of Schlömilch's.

Definite-integral forms of the remainder can be reduced by viii. or ix. above. Thus, to obtain Roche's form in expanding $f(x+h)$ by integration by parts, let the remainder

$$\equiv \frac{1}{n\,!}\int_0^h v^n f^{n+1}(x+h-v)\,dv$$

and consequently
$$= \frac{1}{n\,!}\int_0^h v^{n-p} f^{n+1}(x+h-v)\,.\,v^p dv$$

by ix.
$$= \frac{(\theta h)^{n-p}}{n!} f^{n+1}\{x+(1-\theta)h\}\int_0^h v^p dv$$

$$= \frac{\theta^{n-p}h^{n+1}}{n!\,(p+1)} f^{n+1}\{x+(1-\theta)h\}.$$

More complicated forms of the remainder can be obtained by employing the Extension of Cauchy's Form given in 6. One of these has been noticed by Professor Mansion, *Mess. of Math.*, V_2, 162.

9. *Remainder in Cayley's Theorem.* In the note preceding this article, entitled *Simple and Uniform Method of Obtaining Taylor's, Cayley's, and Lagrange's Series*, it is shown that

$$f(x+a\,;m) = f\left(x+a\,;\genfrac{}{}{0pt}{}{m}{1}\right) + \frac{[a]^1}{1!}f'\left(x+a\,;\genfrac{}{}{0pt}{}{m}{2}\right) + \ldots\ldots$$
$$\ldots\ldots + \frac{[a]^{n-1}}{(n-1)!}f^{n-1}\left(x+a\,;\genfrac{}{}{0pt}{}{m}{n}\right) + R \qquad \text{xvi.}$$

in which
$$R \equiv \left[\int_0^a da\right]^{;\,n} f^n(x+a\,;m) \qquad (\delta).$$

by ix.
$$= \left[\int_0^a da\right]^{;\,n-1}\left(f^n\left\{x+\theta'(a\,;n)+a\,;\genfrac{}{}{0pt}{}{m}{n}\right\}\int_0^{a\,;\,n} d\,(a\,;n)\right)$$

$$= \left[\int_0^a da\right]^{;\,n-1}\left(f^n\left\{x+\theta'(a\,;n)+a\,;\genfrac{}{}{0pt}{}{m}{n}\right\}\;[a]^n_{n-1}\right)$$

by ix.,
$$= \left[\int_0^a da\right]^{;\,n-2}\left(\frac{[a]^n_{n-2}}{2\,!}f^n\left\{x+\theta''\theta_n(a\,;n-1)+\alpha_{n-1}+a\,;\genfrac{}{}{0pt}{}{m}{n}\right\}\right)$$

. . . .

. . . .

$$= \frac{[a]^n}{n\,!}f^n\left(x+\alpha\,;n+a\,;\genfrac{}{}{0pt}{}{m}{n}\right). \qquad (\varepsilon).$$

Substitute x for $x+a\,;m$ and θ for $1-\theta$ and the above becomes

$$fx = f(x-a\,;1) + \frac{[a]^1}{1}f'(x-a\,;2) + \ldots\ldots + \frac{[a]^{n-1}}{(n-1)!}f^{n-1}(x-a\,;n)$$
$$+ \frac{[a]^n}{n!}f^n(x-\alpha\,;n) \qquad \text{xvii.}$$

In III. of *An Extension of Taylor's Theorem*, (this Journal, Vol. I, p. 287), the remainder

$$\equiv \left[\int_0^a da\right]^{;n} \left\{1 - \int_0^{a;n+1} d(a;n+1) . \frac{d}{dx}\right\}^{-1} f^n\left(x + a ; \begin{matrix} m \\ n+1 \end{matrix}\right)$$

and by I. of the same note, this

$$\equiv \left[\int_0^a da\right]^{;n} f^n(x + a ; m)$$

which is (δ) given above.

10. *Remainder in Lagrange's Series.* Let x and y be independent variables, $w = x + y\phi w$, $v = x + a\phi v$, $u = x + b\phi u$ and $u_1 = x + \mu\phi u_1$; also let fw, $\frac{dfw}{dy}$, ... $\frac{d^{n+1}w}{dy^{n+1}}$, $\psi(c - y)$, $\psi'(c - y)$ be all continuous from $y = a$ to $y = b$, and $\psi'(c - \mu) \gtrless 0$.

Since
$$\frac{d}{dy}\left\{(\phi w)^n \frac{dfw}{dx}\right\} = \frac{d}{dx}\left\{(\phi w)^{n+1} \frac{dfw}{dx}\right\},$$

if $Fy \equiv fw + \frac{k-y}{1}\phi w \frac{dfw}{dx} + \ldots\ldots + \frac{(k-y)^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi w)^n \frac{dfw}{dx}\right\}$,

then $F'y \equiv \frac{(k-y)^n}{n!}\left(\frac{d}{dx}\right)^n \left\{(\phi w)^{n+1}\frac{dfw}{dx}\right\}$,

and, by iv.,
$$Fb = Fa + \frac{\psi(c-a) - \psi(c-b)}{\psi'(c-\mu)} F'\mu ;$$

or, writing this in full,

$$fu + \frac{k-b}{1}\phi u \frac{dfu}{dx} + \ldots\ldots\ldots + \frac{(k-b)^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi u)^n \frac{dfu}{dx}\right\} =$$
$$fv + \frac{k-a}{1}\phi v \frac{dfv}{dx} + \ldots\ldots\ldots + \frac{(k-a)^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi v)^n \frac{dfv}{dx}\right\}$$
$$+ \frac{\psi(c-a) - \psi(c-b)}{\psi'(c-\mu)} . \frac{(k-\mu)^n}{n!}\left(\frac{d}{dx}\right)^n \left\{(\phi u_1)^{n+1}\frac{dfu_1}{dx}\right\}. \qquad \text{xviii.}$$

If $k = b$, $c = b + h$ and $a = 0$ and therefore $v = x$ and $u_1 = x + \theta b\phi u_1$, this reduces to Lagrange's Series, or

$$fu = fx + \frac{b}{1}\phi x . f'x + \ldots\ldots\ldots\ldots + \frac{b^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\{(\phi x)^n f'x\}$$
$$+ \frac{\psi(h+b) - \psi h}{\psi'\{h + (1-\theta)b\}} . \frac{\{(1-\theta)b\}^n}{n!}\left(\frac{d}{dx}\right)^n \left\{(\phi u_1)^{n+1}\frac{dfu_1}{dx}\right\}. \qquad \text{xix.}$$

Let $h = 0$ and $\psi b = b^{p+1}$, then the remainder will become

$$\frac{(1-\theta)^{n-p} b^{n+1}}{n!\,(p+1)} \left(\frac{d}{dx}\right)^n \left\{ (\phi u_1)^{n+1} \frac{dfu_1}{dx} \right\}. \qquad \text{xx.}$$

For $\psi\,(c-y)$ write $\psi_1(y+h)$, which may be done since ψ and ψ_1 are both arbitrary, then making $k = b$ and $a = 0$, xviii. becomes

$$fu = fx + \;.\;\;.\;\;. + \frac{\psi_1(c+b) - \psi_1 c}{\psi'_1(c+\theta b)} \cdot \frac{\{(1-\theta)\,b\}^n}{n!} \left(\frac{d}{dx}\right)^n \left\{ (\phi u_1)^{n+1} \frac{dfu_1}{dx} \right\}. \qquad \text{xxi.}$$

Again for $\psi(c-y)$ and $\psi_1(y+h)$, there may be substituted functions of the form of W_x, V_x, or $V_x - W_x$ of 5, or functions bearing the same relation to Lagrange's Series that these do to Taylor's.

In the note *Simple and Uniform Method*, &c., it is shown that if $u = x + b\phi u$ and $v = x + (b-\beta)\phi v$ the remainder in Lagrange's Series is

$$\frac{1}{n!}\int_0^b d\beta \left[\beta^n \left(\frac{d}{dx}\right)^n \left\{ (\phi v)^{n+1} \frac{dfv}{dx} \right\} \right]$$

which
$$= \frac{1}{n!}\int_0^b d\beta \left[\beta^{n-p} \left(\frac{d}{dx}\right)^n \left\{ (\phi v)^{n+1} \frac{dfv}{dx} \right\} \beta^p \right]$$

by ix.
$$= \frac{(1-\theta)^{n-p} b^{n+1}}{n!\,(p+1)} \left(\frac{d}{dx}\right)^n \left\{ (\phi u_1)^{n+1} \frac{dfu_1}{dx} \right\} \qquad \text{xx}_1.$$

in which $u_1 = x + \theta b \phi u_1$. This is the same form of remainder as that found in xx. Had viii. been used instead of ix. in the reduction, forms like those of xix. or xxi. would have been obtained.

II. FUNCTIONS OF A COMPLEX VARIABLE.

11. *General Form.* Let $u = x + i\phi x$, $u_0 = x_0 + i\phi x_0$, $u_1 = x_1 + i\phi x_1$, $v_1 = u_0 + \theta_1(u_1 - u_0)$, and $v_2 = u_0 + \theta_2(u_1 - u_0)$. If ϕx and $\phi' x$ remain continuous while x varies continuously from $x = x_0$ to $x = x_1$, if also Φu and $\Phi' u$ remain continuous from $u = u_0$ to $u = u_1$, and if $\Phi u_1 - \Phi u_0 = 0$, then will

$$\mathfrak{S}\Phi' v_1 = 0 \quad \text{and} \quad \mathfrak{V}\Phi' v_2 = 0. \qquad \text{xxii.}$$

12. *Special Form.* Let $w = (u - u_0) : (u_1 - u_0)$ and

$$\Phi u \equiv (u_1 - u_0)\,\{(Fu_1 - Fu_0)(\Psi w - \Psi 0) - (Fu - Fu_0)(\Psi 1 - \Psi 0)\}$$

and $\therefore\; \Phi' u \equiv (Fu_1 - Fu_0)\,\Psi' w - (u_1 - u_0)(\Psi 1 - \Psi 0)\,F' u$

$$\therefore\; Fu_1 - Fu_0 = \frac{\Psi_1 1 - \Psi_1 0}{\Psi'_1 \theta_1}\, \mathfrak{S}\,\{(u_1 - u_0)\,F' v_1\} + \frac{\Psi_2 1 - \Psi_2 0}{\Psi'_2 \theta_2}\, \mathfrak{V}\{(u_1 - u_0)\,F' v_2\}. \qquad \text{xxiii.}$$

13. *Remainder in Taylor's Thoerem.* In the preceding form let

$$Fu \equiv f(h+lu) + \frac{k-lu}{1} f'(h+lu) + \ldots\ldots\ldots + \frac{(k-lu)^n}{n!} f^n(h+lu),$$

$$\Psi w \equiv \psi(a+bw) + \frac{c-bw}{1}\psi'(a+bw) + \ldots\ldots\ldots + \frac{(c-bw)^m}{m!}\psi^m(a+bw),$$

$$\begin{aligned}
\therefore\ & f(h+lu_1) + \frac{k-lu_1}{1} f'(h+lu_1) + \ldots\ldots\ldots + \frac{(k-lu_1)^n}{n!} f^n(h+lu_1) \\
& - \left\{ f(h+lu_0) + \frac{k-lu_0}{1} f'(h+lu_0) + \ldots\ldots\ldots + \frac{(k-lu_0)^n}{n!} f^n(h+lu_0) \right\} \\
= & \Big[\psi_1(a+b) + \frac{c-b}{1}\psi_1'(a+b) + \ldots\ldots\ldots + \frac{(c-b)^p}{p!}\psi_1^p(a+b) \\
& \qquad - \left\{ \psi_1 a + \frac{c}{1}\psi_1' a + \ldots\ldots\ldots + \frac{c^p}{p!}\psi_1^p a \right\} \Big] R_1 \\
+ & \Big[\psi_2(\alpha+\beta) + \frac{\gamma-\beta}{1}\psi_2'(\alpha+\beta) + \ldots\ldots\ldots + \frac{(\gamma-\beta)^q}{q!}\psi_2^q(\alpha+\beta) \\
& \qquad - \left\{ \psi_2\alpha + \frac{\gamma}{1}\psi_2'\alpha + \ldots\ldots\ldots + \frac{\gamma^q}{q!}\psi_2^q\alpha \right\} \Big] R_2 \equiv R. \qquad \text{xxiv.}
\end{aligned}$$

in which $$R_1 \equiv \frac{p!\, l\, \mathfrak{S}\{(u_1-u_0)(k-lv_1)^n f^{n+1}(h+lv_1)\}}{n!\, b\,(c-\theta_1 b)^p\, \psi_1^{p+1}(a+\theta_1 b)}, \qquad (\zeta_1).$$

and $$R_2 \equiv \frac{q!\, l\, \mathfrak{V}\{(u_1-u_0)(k-lv_2)^n f^{n+1}(h+lv_2)\}}{n!\, \beta\,(\gamma-\theta_2\beta)^q\, \psi_2^{q+1}(\alpha+\theta_2\beta)}. \qquad (\zeta_2).$$

Had it been assumed that

$$\Psi w \equiv \Psi_1(a_1+b_1w) - \Psi_2(a_2-b_2w) + \frac{c_1-b_1w}{1}\Psi_1'(a_1+b_1w) - \frac{c_2-b_2w}{1}\Psi_2'(a_2-b_2w) + \&\text{c.},$$

a general theorem would have been obtained of which v. is the form for a real variable in the particular case $l = b_1 = 1$.

Taylor's theorem with Remainder is the particular case of xxiv. for $h = 0$, $k = u_1$, and $l = 1$. If in addition to these limitations, $c = b$, $\gamma = \beta$, and $u_1 - u_0 = t$, xxiv. becomes (writing u for u_0)

$$\begin{aligned}
& f(u+t) - \left\{ fu + \frac{t}{1} f'u + \ldots\ldots\ldots + \frac{t^n}{n!} f^n u \right\} \\
= & \Big[\psi_1(a+b) - \left\{ \psi_1 a + \frac{b}{1}\psi'_1 a + \ldots\ldots\ldots + \frac{b^p}{p!}\psi_1^p a \right\} \Big] R_1 \\
+ & \Big[\psi_2(\alpha+\beta) - \left\{ \psi_2\alpha + \frac{\beta}{1}\psi'_2 + \alpha \ldots\ldots\ldots + \frac{\beta^q}{q!}\psi_2^q \alpha \right\} \Big] R_2 \qquad \text{xxv.}
\end{aligned}$$

in which
$$R_1 \equiv \frac{p\,!\,(1-\theta_1)^{n-p}\,\mathfrak{S}\{t^{n+1}f^{n+1}(u+\theta_1 t)\}}{n\,!\,b^{p+1}\psi_1{}^{p+1}(a+\theta_1 b)}\,, \qquad (\eta_1).$$

and
$$R_2 \equiv \frac{q\,!\,(1-\theta_2)^{n-q}\,\mathfrak{V}\{t^{n+1}f^{n+1}(u+\theta_2 t)\}}{n\,!\,\beta^{q+1}\,\psi_2{}^{q+2}(\alpha+\theta_1\beta)}\,. \qquad (\eta_2).$$

See *Elementary Demonstration of Taylor's Theorem for Functions of an Imaginary Variable*, by Professor Mansion, *Mess. of Math.* VIII_2. pp. 17–20 (1878).

In this note I have spoken of *Forms of Rolle's Theorem* because in reality the various theorems differ merely in form, not at all in generality. Some forms are more convenient than others, and exhibit explicitly what the others contain implicitly. Thus Cauchy's Form is not more general than that of Lagrange, but only sometimes more convenient; every theorem that can be proved by the former can likewise be proved by the latter.